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POTENTIAL VORTICITY RINGS AND EYE SUBSIDENCE

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1. INTRODUCTION

When a tropical cyclone develops a circularly symmetric eyewall ring of convection, the potential vorticity (PV) in this ring can rapidly increase since the material derivative of PV is proportional to the PV itself. However, if the developing PV ring is narrow enough, it can be dynamically unstable, with a subsequent mixing of high PV from the eyewall into the eye. The resulting decrease of PV in the eyewall can act as a transient "intensification brake." However, in the long run, PV mixing into the eye will lead to larger tangential winds than would occur in the absence of such mixing. These effects have been studied (Kossin et al. 2006) with a forced, nondivergent barotropic model, a context which most simply illustrates the dual nature of PV mixing.

PV mixing implies a change in the inertial stability field, with subsequent effects on the transverse circulation. For example, the upward mass flux across isobaric surfaces in the eyewall is compensated by downward mass fluxes in the eye and in the far-field environment. The percentage of the downward flux that occurs in the eye is determined by geometrical parameters and by the inertial stability distribution. The purpose of the present study is to use the Eliassen transverse circulation equation to quantify this effect and thereby obtain a better understanding of secondary circulation changes that are associated with PV mixing events. The balanced vortex model is presented in section 2. In section 3 idealized solutions of the Eliassen transverse circulation equation are derived. These solutions illustrate how the downward mass flux in the eye depends on the eyewall geometry and the radial distribution of inertial stability.

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2. BALANCED VORTEX MODEL

We consider inviscid, axisymmetric, quasi-static, gradient-balanced motions of a stratified, compressible atmosphere on an f-plane. As the vertical coordinate we use $z = H\ln(p_0/p)$, where $H = RT_0/g$ is the constant scale height, and where p_0 and T_0 are constant reference values of pressure and temperature. We choose $p_0 = 100$ kPa and $T_0 = 300$K, the latter of which yields $H \approx 8.79$ km. The governing equations for the balanced vortex model are

$$\left(f + \frac{v}{r}\right) v = \frac{\partial \phi}{\partial r}, \tag{1}$$

$$\frac{Dv}{Dt} + \left(f + \frac{v}{r}\right) u = 0, \tag{2}$$

$$\frac{\partial \phi}{\partial z} = \frac{g}{T_0} T, \tag{3}$$

$$\frac{\partial (ru)}{r\partial r} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0, \tag{4}$$

$$c_p \frac{DT}{Dt} + \frac{RT}{H} w = Q, \tag{5}$$

where u and v are the radial and azimuthal components of velocity, $w = Dz/Dt$ the "log-pressure vertical velocity," ϕ the geopotential, f the constant Coriolis parameter, Q the diabatic heating, and $D/Dt = \partial/\partial t + u(\partial/\partial r) + w(\partial/\partial z)$ the material derivative.

The potential vorticity equation, derived from (2), (4), and (5), is

$$\frac{DP}{Dt} = e^{z/H} \left[-\frac{\partial v}{\partial z}\frac{\partial \dot{\theta}}{\partial r} + \left(f + \frac{\partial (rv)}{r\partial r}\right)\frac{\partial \dot{\theta}}{\partial z} \right], \tag{6}$$

where

$$P = e^{z/H} \left[-\frac{\partial v}{\partial z}\frac{\partial \theta}{\partial r} + \left(f + \frac{\partial (rv)}{r\partial r}\right)\frac{\partial \theta}{\partial z} \right] \tag{7}$$

is the potential vorticity and $\dot{\theta} = \exp(\kappa z/H)\, Q/c_p$.

The thermal wind equation, derived from (1) and (3), is

$$\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial z} = \frac{g}{T_0}\frac{\partial T}{\partial r}. \tag{8}$$

Taking $\partial/\partial t$ of (8) we obtain

$$\begin{aligned} \frac{g}{T_0}\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial t}\right) &= \frac{\partial}{\partial t}\left[\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial z}\right] \\ &= \frac{\partial}{\partial z}\left[\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial t}\right], \end{aligned} \tag{9}$$

which shows that the tendencies $\partial T/\partial t$ and $\partial v/\partial t$ are related by the constraint of continuous thermal wind balance. The diagnostic equation for the transverse circulation is obtained by eliminating the local time derivatives in (2) and (5) through the use of (9). To accomplish this we first note that, because of the continuity equation (4), the transverse circulation (u, w) can be expressed in terms of the single streamfunction variable ψ such that $e^{-z/H}u = -\partial\psi/\partial z$ and $e^{-z/H}w = \partial(r\psi)/r\partial r$. We next multiply (2) by $-[f + (2v/r)]$ and (5) by g/T_0, and write the resulting equations as

$$-\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial t} + B\frac{\partial(r\psi)}{r\partial r} + C\frac{\partial\psi}{\partial z} = 0, \tag{10}$$

$$\frac{g}{T_0}\frac{\partial T}{\partial t} + A\frac{\partial(r\psi)}{r\partial r} + B\frac{\partial\psi}{\partial z} = \frac{g}{c_p T_0}Q, \tag{11}$$

where the static stability A, the baroclinity B, and the inertial stability C are defined below in (13)–(15). Adding $\partial/\partial r$ of (11) to $\partial/\partial z$ of (10), and then using (9), we obtain the transverse circulation equation

$$\begin{aligned} \frac{\partial}{\partial r}\left(A\frac{\partial(r\psi)}{r\partial r} + B\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(B\frac{\partial(r\psi)}{r\partial r} + C\frac{\partial\psi}{\partial z}\right) \\ = \frac{g}{c_p T_0}\frac{\partial Q}{\partial r}, \end{aligned} \tag{12}$$

where

$$A = e^{z/H}\frac{g}{T_0}\left(\frac{\partial T}{\partial z} + \frac{\kappa T}{H}\right), \tag{13}$$

$$B = -e^{z/H}\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial z} = -e^{z/H}\frac{g}{T_0}\frac{\partial T}{\partial r}, \tag{14}$$

$$C = e^{z/H}\left(f + \frac{2v}{r}\right)\left(f + \frac{\partial(rv)}{r\partial r}\right). \tag{15}$$

We shall only consider vorticies with $AC - B^2 > 0$ everywhere, in which case (12) is an elliptic equation. As for boundary conditions on (12), we require that ψ vanish at $r = 0$ and at the bottom and top isobaric surfaces $z = 0, z_T$, and that $r\psi \to 0$ as $r \to \infty$. In the next section we solve a simplified version of (12) under these boundary conditions.

During a PV mixing event the spatial distribution of $AC - B^2$ can change significantly, with most of the change being due to a change of the inertial stability C. This is more easily understood by noting the connection between $AC - B^2$ and P. This connection is easily derived from (7) and (13)–(15). It takes the form

$$AC - B^2 = \frac{g}{T_0}\exp\left(\frac{z}{\gamma H}\right)\left(f + \frac{2v}{r}\right)P. \tag{16}$$

3. SOLUTIONS OF THE TRANSVERSE CIRCULATION EQUATION

Consider a barotropic vortex ($B = 0$) with static stability given by $A = e^{z/H}N^2$, where the square of the Brunt-Väisälä frequency, N^2, is a constant. The inertial stability (15) can then be written in the form $C = e^{z/H}\hat{f}^2$, where $\hat{f}(r) = \{[f + (2v/r)][f + \partial(rv)/r\partial r]\}^{1/2}$ is the "effective Coriolis parameter." Under the above assumptions, (12) reduces to

$$\begin{aligned} \frac{\partial}{\partial r}\left(\frac{\partial(r\psi)}{r\partial r}\right) + \frac{\hat{f}^2}{N^2}e^{-z/H}\frac{\partial}{\partial z}\left(e^{z/H}\frac{\partial\psi}{\partial z}\right) \\ = \frac{g e^{-z/H}}{c_p T_0 N^2}\frac{\partial Q}{\partial r}. \end{aligned} \tag{17}$$

Here we are particularly interested in the important role played by radial variations of $\hat{f}$.

We now assume that the diabatic heating occurs only in an annular ring, with the particular piecewise constant form

$$Q(r, z) = \exp\left(\frac{z}{2H}\right)\sin\left(\frac{\pi z}{z_T}\right)\begin{cases} 0 & 0 \le r < a, \\ Q_0 & a < r < b, \\ 0 & b < r < \infty, \end{cases} \tag{18}$$

where a, b, and Q_0 are constants. Assuming that $\psi(r,z)$ has the separable form

$$\psi(r,z) = \Psi(r) \exp\left(-\frac{z}{2H}\right) \sin\left(\frac{\pi z}{z_T}\right), \qquad (19)$$

the partial differential equation (17) reduces to the ordinary differential equation

$$r^2 \frac{d^2\Psi}{dr^2} + r\frac{d\Psi}{dr} - \left(\mu^2 r^2 + 1\right)\Psi = 0 \qquad r \neq a, b, \quad (20)$$

where

$$\mu^2 = \frac{\hat{f}^2}{N^2}\left(\frac{\pi^2}{z_T^2} + \frac{1}{4H^2}\right) \qquad (21)$$

is the inverse Rossby length squared. The jump conditions, derived by integrating (17) across narrow intervals straddling the points $r = a$ and $r = b$, are

$$\left[\frac{d(r\Psi)}{r\,dr}\right]_{a-}^{a+} = +\frac{gQ_0}{c_p T_0 N^2} \qquad (22\text{a})$$

and

$$\left[\frac{d(r\Psi)}{r\,dr}\right]_{b-}^{b+} = -\frac{gQ_0}{c_p T_0 N^2}. \qquad (22\text{b})$$

For simplicity we assume that $\mu(r)$, like the diabatic heating, also has the piecewise constant form

$$\mu(r) = \begin{cases} \mu_e & 0 \le r < a, \\ \mu_w & a < r < b, \\ \mu_f & b < r < \infty, \end{cases} \qquad (23)$$

where μ_e, μ_w, μ_f are the constant values of $\mu(r)$ in the eye, eyewall, and far-field respectively. Under the assumption (23), the solution of the ordinary differential equation (20) consists of linear combinations of the modified Bessel functions $I_1(\mu r)$ and $K_1(\mu r)$ in each of the three regions. Because $\Psi = 0$ at $r = 0$, we can discard the $K_1(\mu r)$ solution in the inner region. Similarly, because $r\Psi \to 0$ as $r \to \infty$, we can discard the $I_1(\mu r)$ solution in the outer region. The solution of (20) can then be written as

$$\Psi(r) = \begin{cases} \Psi_a \frac{I_1(\mu_e r)}{I_1(\mu_e a)} & 0 \le r \le a, \\ \Psi_a \frac{\alpha(r,b)}{\alpha(a,b)} + \Psi_b \frac{\alpha(a,r)}{\alpha(a,b)} & a \le r \le b, \\ \Psi_b \frac{K_1(\mu_f r)}{K_1(\mu_f b)} & b \le r < \infty, \end{cases} \qquad (24)$$

where Ψ_a and Ψ_b are constants to be determined by the two jump conditions (22) and

$$\begin{aligned} \alpha(r_1, r_2) = {} & I_1(\mu_w r_1) K_1(\mu_w r_2) \\ & - K_1(\mu_w r_1) I_1(\mu_w r_2). \end{aligned} \qquad (25)$$

Note that (24) has been written in such a way that $\Psi(r)$ is continuous at $r = a$ and $r = b$. Using the derivative relations $d[rI_1(\mu r)]/rdr = \mu I_0(\mu r)$ and $d[rK_1(\mu r)]/rdr = -\mu K_0(\mu r)$ we can differentiate (24) to obtain

$$\frac{d(r\Psi)}{r\,dr} = \begin{cases} \Psi_a \mu_e \frac{I_0(\mu_e r)}{I_1(\mu_e a)} & 0 \le r < a, \\ \Psi_a \mu_w \frac{\beta(r,b)}{\alpha(a,b)} - \Psi_b \mu_w \frac{\beta(r,a)}{\alpha(a,b)} & a < r < b, \\ -\Psi_b \mu_f \frac{K_0(\mu_f r)}{K_1(\mu_f b)} & b < r < \infty. \end{cases} \qquad (26)$$

where

$$\begin{aligned} \beta(r_1, r_2) = {} & I_0(\mu_w r_1) K_1(\mu_w r_2) \\ & + K_0(\mu_w r_1) I_1(\mu_w r_2). \end{aligned} \qquad (27)$$

Use of (26) in the jump conditions (22) leads to two algebraic equations that determine the constants Ψ_a and Ψ_b. With the aid of the Wronskian $I_0(x)K_1(x) + K_0(x)I_1(x) = 1/x$ we can express Ψ_a and Ψ_b as

$$\Psi_a = \frac{gQ_0 \alpha(a,b) b}{c_p T_0 N^2} \left\{ \frac{1 - \mathcal{A}}{1 - \mathcal{A}\mathcal{B}} \right\}, \qquad (28)$$

$$\Psi_b = \frac{gQ_0 \alpha(a,b) a}{c_p T_0 N^2} \left\{ \frac{-1 + \mathcal{B}}{1 - \mathcal{A}\mathcal{B}} \right\}, \qquad (29)$$

where

$$\mathcal{A} = \mu_w a \left(\beta(b,a) - \alpha(a,b) \frac{\mu_f K_0(\mu_f b)}{\mu_w K_1(\mu_f b)} \right), \qquad (30)$$

$$\mathcal{B} = \mu_w b \left(\beta(a,b) - \alpha(a,b) \frac{\mu_e I_0(\mu_e a)}{\mu_w I_1(\mu_e a)} \right). \qquad (31)$$

The upward mass flux in the eyewall is equal to the sum of the downward mass fluxes in the eye and the far-field. From the solutions presented above we can easily calculate the fraction of the downward mass flux that occurs in the eye. On any isobaric surface we have

$$\begin{aligned} \frac{\text{Downward Mass Flux in Eye}}{\text{Total Downward Mass Flux}} &= -\frac{a\Psi_a}{b\Psi_b - a\Psi_a} \\ &= \frac{-1 + \mathcal{A}}{-2 + \mathcal{A} + \mathcal{B}}. \end{aligned} \qquad (32)$$

Note that the five parameters $a, b, \mu_e, \mu_w, \mu_f$ appear on the right hand side of (32). During a PV mixing event the value of μ_e can be significantly increased as the eye becomes more inertially stable. This can change the fraction of the downward mass flux that occurs in the eye. These issues will be discussed in the oral presentation at the conference.

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4. **REFERENCES**

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